†A: Complex Conjugate/Transpose Negation (Orthogonal/Perpendicular Shift).

A ∥ B: Additive Implication (Parallel Processes/Concurrent Choice). \*

A ∦ B: Additive Non-Implication (Non-Parallel/Mutually Exclusive Processes).

¬: Linear Negation (Classical).

⊸: Multiplicative Implication (Linear Implication).

⊸̸: Multiplicative Non-Implication (Type 1A).

▹A: Type 1A Negation (Paracomplete).

◃A: Type 1B Negation (Paraconsistent).

&, ⊕, ⊗, ⅋: Additive and Multiplicative Conjunction/Disjunction.

∃∀ {⊥,⊤,¬,∨,∧,→,↛,↔,⊕,↓,↑} {p, q} ⊢⊬⊨ΓΔ

Signature of Theories with Logos Formalization

The theories discussed in this paper will be referred to as *theories with standard formalization*. They can be briefly characterized as theories which are formalized within the first-order predicate logic (with identity, without variable predicates).

## Syntactic Definitions

### Definition: Variables and Constants of a Theory

The symbols which occur in expressions of a given theory T are divided into *variables* and *constants*.

The set of variables is assumed to be denumerable and hence infinite; the set of constants is either finite or denumerable. All the variables are treated as ranging over the same set of elements.

### 

### Definition: non-Logical Constants of a Theory AKA Signature of a Theory

The non-logical constants are the *predicates* (or *relation symbols*), the *operation symbols*, and the *individual constants*.

With every predicate and every operation symbol a positive integer is correlated which is called the *rank* of the symbol. Thus, we may have in T *unary* predicates and operation symbols (I.E. symbols of rank 1), *binary* predicates and operation symbols (symbols of rank 2), etc. The identity symbol, though regarded as a logical constant, is included in the set of binary predicates.

### Definition: Terms

Among expressions (I.E. finite concatenations of symbols) we distinguish *terms* and *formulas*.

The simplest, so-called atomic, terms are the variables and the individual constants; a compound term is obtained by combining *n* simpler terms by means of an operation symbol of rank *n*.

### Definition: Formulas

Similarly, an atomic formula is obtained by combining *n* arbitrary terms by means of a predicate of rank *n*; compound formulas are built from simpler ones by means of sentential connectives and quantifier expressions (I.E. quantifiers followed by variables like ∀x or ∃y).

### Definition: Sentences

An occurrence of a variable in a formula may be either *free* or *bound*; a formula in which no variable occurs free is called a *sentence*.

## Semantic Definitions

#### Model-Theoretical Semantics

Another method of defining logically derivable and logically valid is available which essentially involves the use of some semantical notions and the notion of satisfaction.

### Definition: Possible Realizations of a Theory and the Universe of 𝕽

We assume that all the non-logical constants of T have been arranged in a (finite or infinite) sequence <**C\_0**, …, **C\_n**, …>, without repeating terms.

We consider systems 𝕽 formed by a non-empty set U and by a sequence <C\_0, …, C\_n, …> of certain mathematical entities, with the same number of terms as the sequence of non-logical constants.

The mathematical nature of each C\_n depends on the logical character of the corresponding constant **C\_n**. Thus,

if **C\_n** is a unary predicate, then C\_n is a subset of U; more generally, if **C\_n** is an *m*-ary predicate, then C\_n is an *m*-ary relation the field of which is a subset of U.

If **C\_n** is an *m*-ary operation symbol, C\_n is an *m*-ary operation (function of *m* arguments) defined over arbitrary ordered *m*-tuples <*x\_1*, …, *x\_m*> of elements of U and assuming elements of U as values.

If **C\_n** is an individual constant, C\_n is simply an element of U.

Such a system (sequence) 𝕽 = <U, C\_0, …, C\_n, …> is called a *possible realization* or simply a *realization* of T; the set U is called the *universe* of 𝕽.

### Definition: Satisfaction

We assume it to be clear under what conditions a sentence Φ of T is said to *be satisfied* or to *hold* in a given realization 𝕽.

Roughly speaking, this means that Φ turns out to be true if

(i) all the variables occurring in T are assumed to range over the set U;

(ii) the logical constants are interpreted in the usual way;

(iii) each of the non-logical constants **C\_n** is understood to denote the corresponding term C\_n in 𝕽.

Assume, e.g., that the term **C\_n** in the sequence of constants is a unary predicate and that consequently C\_n is a subset of U. Then the sentence ∀x **C\_n** x holds in 𝕽 if and only if every element of U is an element of C\_n and hence C\_n coincides with U.

### Definition: Valid Sentences by Non-Logical Axioms

Often we single out a (finite or infinite) set of sentences called *non-logical axioms*, and define a sentence to be valid if and only if it is derivable from this set--or, what amounts to the same, from the set of all axioms, both logical and non-logical.

In all the theories with standard formalization the same symbols are assumed to be used as variables and logical constants; apart from differences in non-logical constants, the same expressions are regarded as formulas, sentences, logical axioms, and logically valid sentences.

However, the notions of validity in these theories may of course exhibit essential differences.

## Theoretical Definitions

### Definition: Uniqueness of an Axiomatic Theory in Standard Formalization

An axiomatic theory is uniquely determined by its non-logical constants and non-logical axioms.

### Definition: Inessential Extensions of Theories in Standard Formalization

An extension T\_2 of T\_1 is called *inessential* if every constant of T\_2 which does not occur in T\_1 is an individual constant and if every valid sentence of T\_2 is derivable in T\_2 from a set of valid sentences of T\_1.

If T\_1 is axiomatic, then an inessential extension of T\_1 is obtained by adding some new individual constants, but without adding any new non-logical axioms.

By saying that a sentence Φ is derivable *in a theory* T from a set A we stress the fact that, in deriving Φ, we may use both sentences of A and logical axioms of T. It is easily seen that, whenever Φ is derivable from A in some theory T, it is also derivable from A in every theory T' which contains all the non-logical constants occurring in Φ and in sentences of A.

### Definition: Union of Theories in Standard Formalization

Among the extensions common to two given theories T\_1 and T\_2 there is always a smallest one, which is a subtheory of any other common extension; this smallest common extension is referred to as the *union* of the given theories.

The union T of T\_1 and T\_2 is fully characterized by the following two conditions:

(i) the set of all non-logical constants of T is the (set theoretical) union of the set of all non-logical constants of T\_1 and T\_2;

(ii) a sentence is valid in T if and only if it is derivable in T from a set of sentences which are valid in T\_1 or T\_2.

If the theories T\_1 and T\_2 are axiomatic, we can construct T by postulating, in addition to (i), the analogous condition for the set of non-logical axioms.